Parchments for CafeOBJ logics

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Joint work with:

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CafeOBJ cube of logics

H = hidden (behavioural)
A = algebra
O = order
M = many
S = sorted

RWL = rewriting logic

HOSA ← HOSRWL
HA ← HRWL
OSA ← OSA
OSRWL ← RWL

MSA ← RWL
• a standard formalization of the concept of the underlying logical system for specification formalisms and most work on foundations of software specification and development from algebraic perspective;

• a formalization of the concept of a logical system for foundational studies:
  – truly abstract model theory
  – proof-theoretic considerations
  – heterogeneous logical environments

using the basics of category theory
Institution: abstraction

\[ \varphi \bullet \]

plus satisfaction relation:

\[ M \models \varphi \]

and so the usual Galois connection between classes of models and sets of sentences, with the standard notions induced:

\[ (Mod[\Phi], Th[\mathcal{M}], Th[\Phi], \Phi \models \varphi, \text{etc}). \]

- Also, possibly adding (sound) consequence: \( \Phi \vdash \varphi \) (implying \( \Phi \models \varphi \)) to deal with proof-theoretic aspects.

Meseguer \( \sim1987 \rightarrow 1989 \)
Institution: first insight

plus satisfaction relation:

\[ M \models_{\Sigma} \varphi \]

and so, for each signature, the usual Galois connection between classes of models and sets of sentences, with the standard notions induced (\( Mod_{\Sigma}[\Phi] \), \( Th_{\Sigma}[M] \), \( Th_{\Sigma}[\Phi] \), \( \Phi \models_{\Sigma} \varphi \), etc).

- Also, possibly adding (sound) consequence: \( \Phi \vdash_{\Sigma} \varphi \) (implying \( \Phi \models_{\Sigma} \varphi \)) to deal with proof-theoretic aspects.
imposing the satisfaction condition:

\[ M' \models \Sigma \sigma(\varphi) \text{ iff } M'\sigma \models \Sigma \varphi \]

Truth is invariant under change of notation and independent of any additional symbols around.
An *institution* $\mathbf{I} = \langle \text{Sign}, \text{Sen}, \text{Mod}, \models \rangle$ consists of:

- a category $\text{Sign}$ of *signatures*
- a functor $\text{Sen}: \text{Sign} \to \text{Set}$
  - $\text{Sen}(\Sigma)$ is the set of $\Sigma$-*sentences*, for $\Sigma \in |\text{Sign}|$
- a functor $\text{Mod}: \text{Sign}^{\text{op}} \to \text{Class}$
  - $\text{Mod}(\Sigma)$ is the class of $\Sigma$-*models*, for $\Sigma \in |\text{Sign}|$
- for each $\Sigma \in |\text{Sign}|$, $\Sigma$-*satisfaction relation* $\models \Sigma \subseteq \text{Mod}(\Sigma) \times \text{Sen}(\Sigma)$

subject to the *satisfaction condition*:

$$M'|_\sigma \models_\Sigma \varphi \iff M' \models_{\Sigma'} \sigma(\varphi)$$

where $\sigma: \Sigma \to \Sigma'$ in $\text{Sign}$, $M' \in \text{Mod}(\Sigma')$, $\varphi \in \text{Sen}(\Sigma)$, $M'|_\sigma$ stands for $\text{Mod}(\sigma)(M')$, and $\sigma(\varphi)$ for $\text{Sen}(\sigma)(\varphi)$. 
Examples abound

• **typical:**
  – **classics:** equational logic; first-order logic (with predicates and equality); higher-order logic; also with partial operations
  – **logics of CafeOBJ:** many-sorted equational logic; order-sorted equational logic; many-sorted logic of rewritings; order-sorted logic of rewritings; behavioural (hidden) many-sorted equational logic; behavioural (hidden) order-sorted equational logic; behavioural (hidden) many-sorted logic of rewritings; behavioural (hidden) order-sorted logic of rewritings.

• **not so typical:**
  – modal logics; logics of constraints; three-valued logics
  – programming language semantics

• **perhaps unexpected:**
  – no sentences; no models; no signatures; trivial satisfaction relations
  – sets of sentences as sentences; sets of sentences as signatures; classes of models as sentences; sets of sentences as models
Our starting point

- **Trivial algebraic institution:**

\[ A = \langle \text{AlgSig}, \text{Sen}^\emptyset, \text{Alg}, \models^\emptyset \rangle \]

many-sorted signatures \( \Sigma \), no sentences, and \( \Sigma \)-algebras \( A \)

- **Ground equational institution:**

\[ \text{GMSA} = \langle \text{AlgSig}, \text{GEQ}, \text{Alg}, \models \rangle \]

many-sorted signatures \( \Sigma \), ground \( \Sigma \)-equations \( t = t' \), \( \Sigma \)-algebras \( A \), and the usual satisfaction \( A \models t = t' \)
• **Ground order-sorted equational institution:**

\[
\text{GOSA} = \langle \text{OSSig}, \text{GOSEQ}, \text{OSAlg}, \models \rangle
\]

order-sorted signatures \( \langle \Sigma, \leq \rangle \), ground \( \langle \Sigma, \leq \rangle \)-equations \( t = t' \) (terms with subsort inclusions and retracts), \( \langle \Sigma, \leq \rangle \)-algebras \( A \), and the usual satisfaction \( A \models t = t' \) with partial evaluation of terms

• **Ground rewriting institution:**

\[
\text{GPRWL} = \langle \text{AlgSig}, \text{GRW}, \text{RAlg}, \models \rangle
\]

many-sorted signatures \( \Sigma \), ground \( \Sigma \)-rewritings \( t \Rightarrow t' \), rewriting (or: preordered) \( \Sigma \)-algebras \( \langle A, \leq \rangle \), and satisfaction \( \langle A, \leq \rangle \models t \Rightarrow t' \) that interprets \( \Rightarrow \) as \( \leq \)

• **Ground behavioural equational institution:**

\[
\text{GHA} = \langle \text{BehSig}, \text{GBEQ}, \text{Alg}, \models \rangle
\]

behavioural signatures \( \langle \Sigma, OBS \rangle \), ground behavioural \( \Sigma \)-equations \( t \sim t' \), \( \Sigma \)-algebras \( A \), and satisfaction \( A \models t \sim t' \) that interprets \( \sim \) as \( OBS \)-indistinguishability
CafeOBJ cube of ground logics

G = ground
H = hidden (behavioural)
A = algebra
O = order
M = many
S = sorted
RWL = rewriting logic
Institution morphism: \( \mu: I \rightarrow I' \)

with the satisfaction condition lurking again:

\[ M \models \mu(\varphi) \text{ iff } \mu(M) \models' \varphi' \]
An *institution morphism* $\mu : \text{I} \rightarrow \text{I}'$ consists of:

- a functor $\mu : \text{Sign} \rightarrow \text{Sign}'$,
- a natural transformation $\mu : \text{Mod} \rightarrow (\mu)^{\text{op}};\text{Mod}'$, and
- a natural transformation $\mu : \mu;\text{Sen}' \rightarrow \text{Sen}$

subject to the following *satisfaction condition*:

$$M \models_{\Sigma} \mu(\varphi) \text{ iff } \mu(M) \models_{\mu(\Sigma)} \varphi'$$

where $\Sigma \in |\text{Sign}|$, $M \in \text{Mod}(\Sigma)$ and $\varphi' \in \text{Sen}'(\mu(\Sigma))$.

With straightforward component-wise composition this yields

the category of institutions and their morphisms

$\text{INS}$
Sample institution morphisms

- trivial morphisms: $\text{GMSA} \rightarrow A$, $\text{GOSA} \rightarrow A$, $\text{GPRWL} \rightarrow A$, $\text{GHA} \rightarrow A$
- $\text{GOSA} \rightarrow A$ easily extends to $\text{GOSA} \rightarrow \text{GMSA}$
- $\text{GHA} \rightarrow A$ does not extend to $\text{GHA} \rightarrow \text{GMSA}$
- $\text{GHA} \rightarrow \text{GMSA}$, given by mapping $\langle\langle S, \Omega \rangle, OBS \rangle \mapsto \langle OBS, \Omega_{OBS} \rangle$
Fact: The category \text{INS} of institutions and institution morphisms is complete.

Proof: To build the limit of a diagram of institutions:

- **Category of signatures**: the limit of the categories of signatures of the institutions in the institution diagram; the resulting signatures "combine" signatures from the institutions in the diagram.

- **For each of the resulting signatures**:
  - **the set of sentences**: the colimit of sets of sentences over individual signatures it combines, linked by sentence translations in the institution diagram.
  - **the class of models**: the limit of classes of models over individual signatures it combines, linked by model translations in the institution diagram.
  - **satisfaction relation**: given uniquely so that the satisfaction condition holds for institution projection morphisms.

- **For each signature morphism**, the translations of sentences and models: given by the colimit/limit properties.
• Limits in $\text{INS}$: a rudimentary way of combining institutions linked by institution morphisms to capture how one institution is built over another.

• This is in contrast with the *Grothendieck institution* built over the same diagram, which just puts the institutions involved next to each other, with additional signature morphisms induced by institution morphisms.
GRWL is the pullback of GMSA and GPRWL over A

Other pullbacks do not give the expected results!
E.g.: the pullback of GOSA and GRWL does not have rewritings between terms with subsort inclusions or retracts

No feature interleaving
We will work with a “model-theoretic” version (rather than relying on super-large “universal” signatures and semantic structures).
A *model-theoretic parchment* \( P = \langle \text{Sign}, \text{Mod}, \text{L}, \mathcal{G} \rangle \) consists of

- a category \( \text{Sign} \) of signatures,
- a functor \( \text{Mod} : \text{Sign}^{\text{op}} \rightarrow \text{Class} \) (as for institutions),
- a functor \( \text{L} : \text{Sign} \rightarrow \text{AlgSig}_* \) giving the abstract syntax of sentences,
- for \( \Sigma \in |\text{Sign}| \) and \( M \in \text{Mod}(\Sigma) \), an \( \text{L}(\Sigma) \)-structure \( \mathcal{G}_{\Sigma}(M) \in |\text{Str}(\text{L}(\Sigma))| \) which determines semantics for \( \Sigma \)-syntax (in particular: semantic evaluation of \( \Sigma \)-sentences in \( M \))
- for \( \sigma : \Sigma \rightarrow \Sigma' \) and \( M' \in \text{Mod}(\Sigma') \), an \( \text{L}(\Sigma) \)-homomorphism \( \mathcal{G}_{\sigma}(M') : \mathcal{G}_{\Sigma}(M'|_{\sigma}) \rightarrow \mathcal{G}_{\Sigma'}(M'|_{\text{L}(\sigma)}) \), compositional in \( \sigma \), to capture uniformity of the semantics

where: \( \text{AlgSig}_* \) is the category of many-sorted signatures with a distinguished “logical” sort \( * \) and a predicate \( D \) on \( * \).
Then:

- \( \text{Sen}(\Sigma) = |T_{L(\Sigma)}|^* \)
- \( M \models_{\Sigma} \varphi \Leftrightarrow D_{g_{\Sigma}(M)}(\varphi_{g_{\Sigma}(M)}) \)
To ensure the satisfaction condition, require parchments to be institutional:

\[ G_\sigma(M') : G_\Sigma(M'|_\sigma) \rightarrow G_{\Sigma'}(M'|_L(\sigma)) \]

preserves and reflects \( D: * \).
From institutional parchments to institutions

Institutional parchment $P$ presents institution $\mathcal{J}(P)$

$$P = \langle \text{Sign}, \text{Mod}, L, G \rangle \text{ presents } \mathcal{J}(P) = \langle \text{Sign}, \text{Sen}, \text{Mod}, \models \rangle$$

where:

- **Sign** is inherited from $P$
- **Sen** is $|T_L(\_)|^*$, so that for $\Sigma \in |\text{Sign}|$, $\text{Sen}(\Sigma) = |T_L(\Sigma)|^*$
- **Mod** is inherited from $P$
- $M \models^\Sigma \varphi$ iff $D g^\Sigma(M)(\varphi g^\Sigma(M))$, for $\Sigma \in |\text{Sign}|$, $M \in \text{Mod}(\Sigma)$, $\varphi \in |T_L(\Sigma)|^*$

Satisfaction condition holds if $P$ is institutional
Our starting point

- \( P_A = \langle \text{AlgSig}, L^A, \text{Alg}, G^A \rangle \) presents \( A = \mathcal{J}(P_A) \):
  
  \( L^A(\Sigma) \) extends \( \Sigma \) with \( * \) and \( D : * \)

- \( P_{\text{GMSA}} = \langle \text{AlgSig}, L^{G\text{MSA}}, \text{Alg}, G^{G\text{MSA}} \rangle \) presents \( \text{GMSA} = \mathcal{J}(P_{\text{GMSA}}) \):
  
  \( L^{G\text{MSA}}(\Sigma) \) extends \( L^A(\Sigma) \) with \( eq : s \times s \to * \) for each sort \( s \), interpreted as the diagonal in \( G^{G\text{MSA}}(A) \)

- \( P_{\text{GOSA}} = \langle \text{OSSig}, L^{G\text{OSA}}, \text{OSAlg}, G^{G\text{OSA}} \rangle \) presents \( \text{GOSA} = \mathcal{J}(P_{\text{GOSA}}) \):
  
  \( L^{G\text{OSA}}(\langle \Sigma, \leq \rangle) \) extends \( L^A(\Sigma) \) with subsort inclusions and retracts, as well as \( eq : s \times s \to * \) for each sort \( s \), and \( G^{G\text{OSA}}(\langle \Sigma, \leq \rangle)(A) \) adds to \( A \) new values \( \bot \) and interprets the new operations as expected

- \( P_{\text{GPRWL}} = \langle \text{AlgSig}, L^{G\text{PRWL}}, \text{RAlg}, G^{G\text{PRWL}} \rangle \) presents \( \text{GPRWL} = \mathcal{J}(P_{\text{GPRWL}}) \):
  
  \( L^{G\text{PRWL}}(\Sigma) \) extends \( L^A(\Sigma) \) with \( rwrt : s \times s \to * \) for each sort \( s \), interpreted as \( \preceq \) in \( G^{G\text{PRWL}}(\langle A, \preceq \rangle) \)

- \( P_{\text{GHA}} = \langle \text{BehSig}, L^{G\text{HA}}, \text{Alg}, G^{G\text{HA}} \rangle \) presents \( \text{GHA} = \mathcal{J}(P_{\text{GHA}}) \):
  
  \( L^{G\text{HA}}(\langle \Sigma, \text{OBS} \rangle) \) extends \( L^A(\Sigma) \) with \( beq : s \times s \to * \) for each sort \( s \), interpreted as \( \text{OBS} \)-indistinguishability in \( G^{G\text{HA}}(\langle \Sigma, \text{OBS} \rangle)(A) \)
example

\[ P_{\text{GOSA}} = \langle \text{OSSig}, L^{\text{GOSA}}, \text{OSAlg}, G^{\text{GOSA}} \rangle \] presents \( \text{GOSA} = \mathcal{J}(P_{\text{GOSA}}) \):

Syntax: for any order-sorted signature \( \langle \Sigma, \leq \rangle \), \( L^{\text{OSA}}(\langle \Sigma, \leq \rangle) \) consists of
- sort * with predicate \( D : * \)
- all sort and operation names from \( \Sigma \)
- for any sort names \( s \leq s' \) in \( \Sigma \), operation names for subsort inclusion and retract \( i : s \to s' \) and \( r : s' \to s \)
- for any sort name \( s \) in \( \Sigma \), operation name for equality \( eq : s \times s \to * \)

Semantics: for any order-sorted signature \( \langle \Sigma, \leq \rangle \) and order-sorted algebra \( A \in \text{OSAlg}(\langle \Sigma, \leq \rangle) \), the evaluation structure \( G^{\text{GOSA}}_{\langle \Sigma, \leq \rangle}(A) \) is given as follows
- the carrier of sort * is \( \text{Bool} = \{ tt, ff \} \), \( D : * \) holds for \( tt \), as usual
- the carriers for sort names in \( \Sigma \) are the corresponding carriers of \( A \) with a new value \( \perp \) added
- the operations from \( \Sigma \) are interpreted as in \( A \), extended to preserve \( \perp \)
- subsort inclusions are interpreted as inclusions and retracts as partial projections, yielding \( \perp \) for extra values in the supersort carriers
- equalities \( eq \) yield \( tt \) for equal arguments in \( |A| \), and \( ff \) otherwise
A (model-theoretic) *parchment morphism* $\gamma: P \rightarrow P'$ consists of:

- a functor $\gamma: \text{Sign} \rightarrow \text{Sign}'$,
- a natural transformation $\gamma: \text{Mod} \rightarrow (\gamma)^{op};\text{Mod}'$,
- a natural transformation $\gamma: \gamma;L' \rightarrow L$, and
- for $\Sigma \in |\text{Sign}|$ and $M \in \text{Mod}(\Sigma)$, an $L'(\gamma(\Sigma))$-homomorphism

$$g_{\Sigma,M}: G_{\gamma(\Sigma)}(\gamma\Sigma(M)) \rightarrow G_{\Sigma}(M)\big|_{\gamma\Sigma}$$

subject to the routine (though not easy) naturality condition:

for $\sigma: \Sigma_1 \rightarrow \Sigma_2$ and $M_2 \in \text{Mod}(\Sigma_2)$,

$$g_{\Sigma_1,M_2}\big|_{\sigma};G_{\sigma}(M_2)\big|_{\gamma\Sigma_1} = G_{\gamma(\sigma)}(\gamma\Sigma_2(M_2));g_{\Sigma_2,M_2}\big|_{L'(\gamma(\sigma)))}$$

(which makes $g$ a 2-natural transformation...).
Parchment morphisms: $\gamma: P \rightarrow P'$

$\gamma$ is *institutional* if $g_{\Sigma,M}: G'_{\gamma(\Sigma)}(\gamma_{\Sigma}(M)) \rightarrow G_{\Sigma}(M)|_{\gamma_{\Sigma}}$ preserves and reflects $D: *$. 
Fact: There is a functor

\[ \mathcal{J} : \text{IPAR} \rightarrow \text{INS} \]

that maps institutional parchments to institutions and institutional parchment morphisms to institution morphisms.

Indeed, institutional parchments smoothly present institutions!

BUT: the construction only “nearly works” for arbitrary (non-institutional) parchments and parchment morphisms — the satisfaction condition may fail.
Sample parchment morphisms

- trivial *institutional* morphisms: $\mathcal{P}_{GMSA} \to \mathcal{P}_{A}$, $\mathcal{P}_{GOSA} \to \mathcal{P}_{A}$, $\mathcal{P}_{GPRWL} \to \mathcal{P}_{A}$, $\mathcal{P}_{GHA} \to \mathcal{P}_{A}$
- $\mathcal{P}_{GOSA} \to \mathcal{P}_{A}$ easily extends to *institutional* $\mathcal{P}_{GOSA} \to \mathcal{P}_{GMSA}$
- $\mathcal{P}_{GHA} \to \mathcal{P}_{A}$ easily extends to *non-institutional* $\mathcal{P}_{GHA} \to \mathcal{P}_{GMSA}$
- *institutional* $\mathcal{P}_{GHA} \to \mathcal{P}_{GMSA}$, given by mapping $\langle\langle S, \Omega \rangle, OBS \rangle \mapsto \langle OBS, \Omega_{OBS} \rangle$

presenting the corresponding diagram of institutions
Fact: The category \( \text{PAR} \) of parchments and their morphisms is complete.

Fact: The category \( \text{IPAR} \) of institutional parchments and their institutional morphisms is not complete.

Fact: Given a diagram of institutional parchments and their institutional morphisms in \( \text{IPAR} \), its limiting cone in \( \text{PAR} \) consists of institutional parchment morphisms, although the limit parchment may be non-institutional.

Fact: Given a diagram of institutional parchments and their institutional morphisms in \( \text{IPAR} \), even if its limit in \( \text{PAR} \) is institutional, it does not have to be its limit in \( \text{IPAR} \).
Putting parchments together

- The pullback of $P_{GMSA}$ and $P_{GPRWL}$ over $P_A$ in $PAR$ is their pullback in $IPAR$ and presents $GRWL$.

- Other pullbacks do not give the expected results!

Problems caused by feature interleaving
Given a diagram of institutional parchments and their institutional morphisms, a cone of institutional morphisms on this diagram is a complete joint extension if each of the logical values in any of the evaluation structures of its vertex parchment corresponds to a logical value in an evaluation structure of a parchment in the diagram.

**Fact:** Given a diagram of institutional parchments and their institutional morphisms in IPAR, if its limiting cone in PAR is a complete joint extension, then it is the limit of this diagram in IPAR.
Adjusting limit parchments

**Typically:** new feature interleavings *freely generate new semantic values*, including those in the sort $\ast$, thus indicating where essential semantic design decisions are necessary for the logic combination

**Technique:** choose *complete coherent family congruences* on the evaluation structures in the limit parchment to glue the new “logical” values with the standard (old) ones; then the quotient of the limit parchment becomes a complete joint extension of the parchments in the diagram

Parchment extension

**Typically:** new features may be freely added to the syntax part of a parchment, resulting in its free extension with *freely generated new semantic values*, including those in the sort $\ast$, thus indicating where essential semantic design decisions are necessary for the logic extension

**Technique:** choose *complete coherent family congruences* on the evaluation structures in the freely extended parchment to glue the new “logical” values with the standard (old) ones; then the quotient of the freely extended parchment becomes a complete (joint) extension of the original parchment
Examples

GOSRWL: The pullback of $P_{GOSA}$ and $P_{GRWL}$ over $P_{GMSA}$ in $PAR$ involves evaluation structures where rewritings between terms that evaluate to $\bot$ generate new logical values. Identifying these new logical values with $ff$ yields a complete coherent family congruences. Put $P_{GOSRWL}$ to be the quotient of the pullback parchment by this family. We thus get a presentation of GOSRWL.

GHRWL: The pullback of $P_{GHA}$ and $P_{GRWL}$ over $P_{GMSA}$ in $PAR$ is in fact a complete joint extension — so it is a combination of the two logics.

BUT: the pullback misses rewritings between terms of non-observable sorts. First, these can be added freely. Then, we can choose a complete coherent family of congruences by gluing the logical values built by behavioural rewritings with $tt$ if the two argument values are in the largest precongruence determined by the observable preorder, and with $ff$ otherwise. Put $P_{GOSRWL}$ to be the quotient of the pullback parchment by this family. We thus get a presentation of GHRWL.
We have now:

- \( P_{GHA} \)
- \( P_{GHRWL} \)
- \( P_{GOSA} \)
- \( P_{GOSRWL} \)
- \( P_{GMSA} \)
- \( P_{GRWL} \)

- **Parchments for the other logics may be constructed similarly...**
We have systematically built:

well, we are “almost” there... many details to be filled in

presenting the corresponding cube of CafeOBJ (ground) institutions